JOURNAL OF APPROXIMATION THEORY 22, 150–160 (1978)

On the Newton–Padé Approximation Problem

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Communicated by Yudell L. Luke

Received January 19, 1976

To study the structure of the Newton–Padé table a new concept—the minimal solution—is introduced. The relationship of the minimal solution with the Newton–Padé approximant is given. A notion of normality, called paranormality, is introduced for the minimal solutions. A paranormal minimal solution is proved to have a characterization analogous to that of a normal Padé approximant.

1. THE NEWTON–PADÉ APPROXIMATION PROBLEM

Let $\{z_i\}_{i=0}^{\infty}$ be a sequence of (not necessarily distinct) points in the complex plane. Let $f(z)$ be a function which is holomorphic on some open set E containing these points: $f(z) \in H(E)$. Then one can construct in a purely formal manner a corresponding interpolation series, also called a Newton series (see, e.g., Walsh [9, p. 53]). This formal interpolation series has the form

$$f \equiv f_{00} + f_{01}(z - z_0) + f_{02}(z - z_0)(z - z_1) + \cdots + f_{0i}(z - z_0)(z - z_1) \cdots (z - z_{i-1}) + \cdots.$$

For abbreviation we put $\omega_{00}(z) = 1$ and $\omega_{0i}(z) = (z - z_{i-1})\, \omega_{0,i-1}(z)$, for $i = 1, 2, \ldots$. Consequently

$$f \equiv \sum_{i=0}^{\infty} f_{0i}\omega_{0i}(z).$$

The coefficients f_{0i} of the $\omega_{0i}(z)$ are divided differences (with possible confluent arguments), i.e. [9, p. 54],

$$f_{0i} = \frac{1}{2\pi i} \int_C \frac{f(t)}{\omega_{0i}(t)}\, dt,$$

where C is a contour or a union of mutually exterior contours belonging to

0021-9045/78/0222-0150$02.00/0

E and containing $z_0, z_1, \ldots, z_i$ in its interior. More generally, by f_{ij} we denote the divided difference of order $j-i$ ($j \geqslant i$), determined by the interpolation points $z_i, z_{i+1}, \ldots, z_j$. If $j < i$, then $f_{ij} = 0$ by convention.

Let $(m, n) \in \mathbb{N}^2$ and $f(z) \in H(E)$; then the Newton–Padé approximation problem for $f(z)$, of order $[m, n]$, can be described as follows: Find two polynomials, $p(z) = \sum_{i=0}^{m} a_{0i}\omega_{0i}(z)$ and $q(z) = \sum_{i=0}^{n} b_{0i}\omega_{0i}(z)$, satisfying

$$\partial p \leqslant m, \qquad \partial q \leqslant n, \tag{1a}$$

$$qf - p = \omega_{0,m+n+1}(z) \cdot v(z), \qquad \text{with} \quad v \in H(E). \tag{1b}$$

Here ∂ stands for "degree of."

Introducing the function σ: $H(E) \to \mathbb{N}$, defined by $\sigma(f) = n$ if and only if $f_{0i} = 0$ for $i = 0, 1, \ldots, n-1$ and $f_{0n} \neq 0$, then (b) is equivalent to $\sigma(qf - p) \geqslant m + n + 1$. Indeed by using the definition, (1b) implies that $(qf - p)_{0i} = 0$ for $i = 0, 1, \ldots, m+n$ and consequently $\sigma(qf - p) \geqslant m + n + 1$. Conversely, if $\sigma(qf - p) \geqslant m + n + 1$, then $(qf - p)_{0i} = 0$ $i = 0, 1, \ldots, m+n$ and we can factor out $\prod_{i=0}^{m+n} (z - z_i) = \omega_{0,m+n+1}$.

Note that the Newton–Padé approximation problem contains the Padé approximation problem as a special case, viz., when $z_i = z_0$ for $i = 1, 2, \ldots$.

Associated with the formal Newton series f we define the generalized Hankel determinants

$$H_{n+1}^{n,m} = \begin{vmatrix} f_{n,m} & f_{n,m+1} & \cdots & f_{n,m+n} \\ f_{n-1,m} & f_{n-1,m+1} & \cdots & f_{n-1,m+n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{0,m} & f_{0,m+1} & \cdots & f_{0,m+n} \end{vmatrix} \qquad \text{with} \quad n, m \in \mathbb{N}.$$

It is easy to verify that for Padé approximations

$$H_{n+1}^{n,m} = H_{n+1}^{m-n},$$

where H_{n+1}^{m-n} denotes the Hankel determinant as defined, e.g., by Henrici [6, p. 594].

2. The Minimal Solution

It is known [4] that the Newton–Padé problem can equivalently be stated as follows. Solve the homegeneous system of equations:

$$\sum_{j=0}^{k} b_{0j} f_{jk} = a_{0k}, \qquad k = 0, 1, \ldots, m, \tag{2a}$$

$$= 0, \qquad k = m+1, m+2, \ldots, m+n, \tag{2b}$$

for the unknowns $a_{00}, a_{01}, \ldots, a_{0m}$ and $b_{00}, b_{01}, \ldots, b_{0n}$. In (2) we use the convention that $b_{0k} = 0$ if $k > n$. By considering the coefficient matrix of (2) it is clear that the rank of this coefficient matrix is completely determinded by the rank of the coefficient matrix of (2b). Therefore if, e.g., the rank of the coefficient matrix of (2b) is $n - d$, then we will say that the rank of the system (2) is $n - d$ and we will use the notation $\operatorname{rank}[m, n] = n - d$.

THEOREM 1. *If rank $[m, n] = n - d$, then there exists a unique solution (except for constant factor) p^*, q^* for* (1) *with $\partial p^* \leqslant m - d$ and $\partial q^* \leqslant n - d$, where at least one of the upper bounds is reached. Every other solution of* (1) *can be written in the form $s(z) \cdot p^*(z)$, $s(z) \cdot q^*(z)$, where $s(z)$ is a polynomial with degree less than or equal to d.*

Proof. First note that a solution of (1) always exists, since (2) is a homogeneous set of $m + n + 1$ equations in $m + n + 2$ unknowns. Since $\operatorname{rank}[m, n] = n - d$, we can construct a solution p_1, q_1 of (1) such that $\partial p_1 \leqslant m$, $\partial q_1 \leqslant n - d$. For the same reason we can also construct a solution p_2, q_2 with $\partial p_2 \leqslant m - d$ and $\partial q_2 \leqslant n$. Then, since

$$p_1 q_2 - p_2 q_1 = q_1(q_2 f - p_2) - q_2(q_1 f - p_1),$$

we have

$$p_1 q_2 - p_2 q_1 = \omega_{0, m+n+1}(z) \cdot v(z), \qquad \text{with} \quad v \in H(E).$$

The left-hand side however is a polynomial of degree at most $m + n$, consequently

$$p_1 q_2 \equiv p_2 q_1 .$$

And since the right-hand side of this expression has degree at most $m + n - 2d$, we must have either $\partial p_1 \leqslant m - d$ or $\partial q_2 \leqslant n - d$. Hence, there exists a solution p^*, q^* of (1) with $\partial p^* \leqslant m - d$ and $\partial q^* \leqslant n - d$. But then, other solutions of (1) are $\omega_{0i} p^*$, $\omega_{0i} q^*$ with $1 \leqslant i \leqslant d$. Since these solutions are linear independent solutions of the system $[m, n]$, they form a basis for the solution space. Consequently, every solution of (1) can be written in the form $s(z) \cdot p^*(z)$, $s(z) \cdot q^*(z)$, where $s(z)$ is a polynomial of degree at most d. This also implies the unicity (except for a constant factor) of the solution p^*, q^* for which $\partial p^* \leqslant m - d$, $\partial q^* \leqslant n - d$. Were the degrees of both p^* and q^* less than their respective upper bounds, then the solution space would have a dimension greater than $d + 1$, which would imply rank $[m, n] < n - d$, which is a contradiction. ∎

Consequently, p^*, q^*, as defined in the theorem, is the solution of (1) of minimal degree. We will call this solution the minimal solution for the system of order $[m, n]$, and we will denote it by p^*_{mn}, q^*_{mn}. Hence the minimal solution is determined except for a constant factor. It will be unique after choosing a certain normalization.

It is known [12, p. 838] that all the rational forms p/q constructed with possible solutions of (1) have the same irreducible form $r_{mn} = p_{mn}/q_{mn}$. This unique rational function is called the Newton–Padé approximant of order $[m, n]$.

THEOREM 2. *If $(z - \alpha)^s$ represents a common factor of p^*_{mn}, q^*_{mn}, then $\alpha \in \{z_i\}_{i=0}^{m+n}$ and $s \leqslant m_\alpha$, where m_α denotes the multiplicity of α in $\{z_i\}_{i=0}^{m+n}$.*

Proof. Suppose p^*_{mn} and q^*_{mn} have a common factor of the form $(z - \beta)^s$, with $\beta \notin \{z_i\}_{i=0}^{m+n}$. Then clearly $(z - \beta)^s$ is a factor of $v(z)$ in

$$q^*_{mn}f - p^*_{mn} = \omega_{0,m+n+1}(z) \cdot v(z), \qquad v \in H(E).$$

Consequently $p^*_{mn}/(z - \beta)^s$, $q^*_{mn}/(z - \beta)^s$ are also solutions of (1). This however is impossible since p^*_{mn}, q^*_{mn} is the minimal solution of (1). On the other hand suppose p^*_{mn}, q^*_{mn} have a common factor of the form $(z - \alpha)^s$ with $\alpha \in \{z_i\}_{i=0}^{m+n}$ and $s > m_\alpha$. Then $v(z)$ must contain at least a factor $(z - \alpha)^{s-m_\alpha}$. And consequently another solution of (1) is given by $p^*_{mn}/(z - \alpha)^{s-m_\alpha}$, $q^*_{mn}/(z - \alpha)^{s-m_\alpha}$, which again contradicts the minimality of p^*_{mn}, q^*_{mn}. ∎

Hence the greatest common divisor $d(z)$ of the minimal solution p^*_{mn}, q^*_{mn} has the form

$$d(z) = \prod_{i=1}^{l} (z - z_{\alpha_i})$$

with $0 \leqslant l \leqslant \min(m, n)$ and with $\{z_{\alpha_i}\}_{i=1}^{l} \subset \{z_i\}_{i=0}^{m+n}$. Here we take as a convention that $d(z) = 1$ if $l = 0$. Then clearly as a consequence of this remark, the following relationship must hold between the minimal solution and the corresponding Newton–Padé approximant,

$$\begin{aligned} p^*_{mn}(z) &= d(z) \cdot p_{mn}(z), \\ q^*_{mn}(z) &= d(z) \cdot q_{mn}(z), \end{aligned} \tag{3}$$

where both the minimal solution and the Newton–Padé approximant are normalized such that q^*_{mn} and q_{mn} are monic polynomials.

EXAMPLE 1. Let $z_i = i - 3$ for $i = 0, 1, \ldots, 4$ and $z_5 = -2$. And let $f(-3) = 1, f(-2) = 2, f(-1) = 1, f(0) = 10, f(1) = 5$, and $f'(-2) = 1$.

Then $p_{32} = z + 4$, $q_{32} = 1$, and

$$q^*_{32} = z(z+1)\,p_{32}\,,$$

$$q^*_{32} = z(z+1)\,q_{32}\,.$$

If $(z - z_\alpha)^s$, $s > 0$, is a common factor of the minimal solution p^*_{mn}, q^*_{mn}, while $(z - z_\alpha)^{s+1}$ is not, then we say that the interpolation point z_α is unattainable for r_{mn}, and that this point z_α has an unattainability of order s. Note that $s \leqslant \min(m, n, m_\alpha)$ where m_α is the multiplicity of z_α in $\{z_i\}_{i=0}^{m+n}$. When a Newton–Padé approximant has unattainable points, we say that it is degenerate. This terminology is explained by the following two theorems, the first of which is given in Wuytack [12, p. 839].

THEOREM 3. *The Newton–Padé approximant* $r_{mn} = p_{mn}/q_{mn}$ *interpolates the function* $f(z) \in H(E)$ *in the points* $\{z_0, z_1, \ldots, z_{m+n}\}$ *if and only if* p_{mn} *and* q_{mn} *satisfy* (1).

Hence, in view of (3), r_{mn} is an interpolant (in the Hermite sense) if and only if $d(z) = 1$. If $d(z) \neq 1$ then $d(z)$ gives information concerning the points in which the interpolation condition is violated. This is shown by the next theorem.

THEOREM 4. *If* $z_{\alpha_1}, z_{\alpha_2}, \ldots, z_{\alpha_l}$ $(0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_l < \alpha_{l+1} = m + n + 1)$ *denote the points of* $\{z_i\}_{i=0}^{m+n}$ *which are equal to* z_α, *then the interpolation point* z_α *has an unattainability of order* s *for* r_{mn} *with* $1 \leqslant s \leqslant l$ *if and only if*

$$r^{(i)}_{mn}(z_\alpha) = f^{(i)}(z_\alpha), \qquad \textit{for} \quad i = 0, 1, \ldots, l - s - 1$$

and

$$r^{(l-s)}_{mn}(z_\alpha) \neq f^{(l-s)}(z_\alpha).$$

The interpolation point z_α *will be attainable if and only if*

$$r^{(i)}_{mn}(z_\alpha) = f^{(i)}(z_\alpha), \qquad \text{for} \quad i = 0, 1, \ldots, l - 1.$$

To prove this theorem we need two auxiliary results, the first of which is due to Salzer [7, p. 487].

LEMMA 1. *Let* $N(z)$ *and* $D(z)$ *be two polynomials. If* $D(\alpha) \neq 0$, *then the system*

$$(N/D)^{(i)}(\alpha) = f^{(i)}(\alpha) \qquad \textit{for} \quad i = 0, 1, \ldots, k,$$

is equivalent to the system

$$N^{(i)}(\alpha) = (Df)^{(i)}(\alpha) \qquad \textit{for} \quad i = 0, 1,..., k.$$

LEMMA 2. *If* $z_{\alpha_1}, z_{\alpha_2},..., z_{\alpha_l}$ $(0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_l < m + n + 1 = \alpha_{l+1})$ *denote the points of* $\{z_i\}_{i=0}^{m+n}$ *which are equal to* z_α, *and if* z_α *has an unattainability of order* s $(1 \leqslant s \leqslant l)$ *for* r_{mn}, *then*

$$\sigma\left(\frac{q_{mn}^* f - p_{mn}^*}{(z - z_\alpha)^s}\right) = \alpha_{l-s+1}.$$

Proof. Since p_{mn}^*, q_{mn}^* is the minimal solution, we have

$$q_{mn}^* f - p_{mn}^* = \omega_{0,m+n+1}(z) \cdot v(z), \qquad v \in H(E).$$

We know $\omega_{0,m+n+1}(z)$ contains exactly l factors $(z - z_\alpha)$. Put $\sigma(q_{mn}^* f - p_{mn}^*/(z - z_\alpha)^s) = S$. Was $S < \alpha_{l-s+1}$, then, necessarily $S \leqslant \alpha_{l-s}$. But this implies that $\sigma(q_{mn}^* f - p_{mn}^*) \leqslant \alpha_l < m + n + 1$, which contradicts the minimality of p_{mn}^* : q_{mn}^*. Assuming $S > \alpha_{l-s+1}$, then $v(z)$ contains at least one factor $z - z_\alpha$. But then $p_{mn}^*/(z - z_\alpha)$, $q_{mn}^*/(z - z_\alpha)$ is also a solution for the Newton–Padé problem of order $[m, n]$, which contradicts the minimality of p_{mn}^*, q_{mn}^*. ∎

Note that this lemma only holds if we work with a prescribed fixed ordering of the interpolation points. Knowing these two results we can proceed with the proof of Theorem 4.

Proof. Suppose z_α has an unattainability of order s for r_{mn}, then in view of Lemma 2,

$$\frac{q_{mn}^* f - p_{mn}^*}{(z - z_\alpha)^s} = (z - z_\alpha)^{l-s} \cdot v(z), \qquad \text{with} \quad v(z_\alpha) \neq 0.$$

Since z_α has an unattainability of order s, $d(z)$ (defined by (3)) contains exactly s factors $(z - z_\alpha)$. Consequently, we have that

$$\frac{q_{mn}^* f - p_{mn}^*}{d(z)} = (z - z_\alpha)^{l-s}\, w(z), \qquad \text{with} \quad w(z) = \frac{v(z) \cdot (z - z_\alpha)^s}{d(z)}$$
$$\text{and} \qquad w(z_\alpha) \neq 0,$$

or

$$q_{mn} f - p_{mn} = (z - z_\alpha)^{l-s} w(z), \qquad \text{with} \quad w(z_\alpha) \neq 0.$$

Hence,

$$p_{mn}^{(i)}(z_\alpha) = (q_{mn} f)^{(i)}(z_\alpha) \qquad \text{for} \quad i = 0, 1,..., l - s - 1,$$

and

$$p_{mn}^{(l-s)}(z_\alpha) \neq (q_{mn}f)^{(l-s)}(z_\alpha).$$

Then, using Lemma 1, the assertion follows. To prove the sufficiency, suppose z_α has an unattainability of order $s' \neq s$. Then applying the first part of the theorem gives a contradiction. The second part of the theorem is proved analogously. ▮

EXAMPLE 2. Let $z_0 = -1$, $z_1 = -\frac{1}{2}$, $z_2 = 0$, $z_3 = \frac{1}{2}$, $z_4 = 1$, $z_5 = -1$, $z_6 = 0$, and $f(-1) = 1$, $f(-\frac{1}{2}) = \frac{1}{2}$, $f(0) = 1$, $f(\frac{1}{2}) = \frac{3}{2}$, $f(1) = 3$, $f'(-1) = -\frac{5}{3}$ and $f'(0) = 2$. Then

$$r_{42} = p_{42}/q_{42} = 1 - (z + 1) + \tfrac{4}{3}(z + 1)(z + \tfrac{1}{2}) = \tfrac{4}{3}z^2 + z + \tfrac{2}{3}$$

and

$$p_{42}^* = z^2 p_{42},$$

$$p_{42}^* = z^2 q_{42}.$$

It is easily verified that $\sigma((q_{42}^* f - p_{42}^*)/z^2) = 2$. The interpolation point $z = 0$ has an unattainability of order 2 for r_{42}, while the other interpolation points are attainable. Hence r_{42} is degenerate. Note also that in Example 1 the interpolation points $z = -1$ and $z = 0$ are both unattainbable for r_{32}.

Both the minimal solutions and the Newton–Padé approximants can be arranged in a two-dimensional array. These tables are called the minimal solution table and the Newton–Padé table, respectively.

3. PARANORMALITY

As for the Padé table the notion of normality is uniquely defined [5, p.16]. A look at the literature makes it clear that this is not the case for the Newton–Padé table. For instance the definition given by Wuytack [11, p. 56] does not agree with Warner's definition [10, p. 39]. Having introduced the concept of minimal solution, it seems natural to introduce a definition of normality for these minimal solutions. However, to distinguish from the existing definitions of normality for the Newton–Padé table, we prefer to call it paranormality.

The minimal solution p_{mn}^*, q_{mn}^* is called paranormal if it occurs only once in the minimal solution table. The corresponding Newton–Padé approximant will then also be called paranormal. If all the elements are paranormal then the minimal solution table and the Newton–Padé table will be called paranormal. As will be proved in the next theorem the notion of paranormality possesses a characterization analogous to the notion of normality in the Padé table.

THEOREM 5. *The following statements are equivalent*:

(a) p_{mn}^*, q_{mn}^* *is paranormal*;

(b) $\partial p_{mn}^* = m$, $\partial q_{mn}^* = n$, *and* $\sigma(q_{mn}^* f - p_{mn}^*) = m + n + 1$;

(c) *the determinants* $H_{n+1}^{n,m}$, $H_n^{n-1,m+1}$, *and* $H_{n+1}^{n,m+1}$ *do not vanish.*

Proof. The theorem will be most easily proved by showing the equivalence of (a) and (b), and of (b) and (c)

(i) (a) implies (b). Suppose $\partial p_{mn}^* < m$. Then p_{mn}^*, q_{mn}^* is also a solution of the system $[m-1, n+1]$. Consequently, since p_{mn}^*, q_{mn}^* is paranormal, there exists a polynomial $d(z)$ with $\partial d > 0$, such that

$$p_{mn}^* = d(z) \cdot p_{m-1,n+1}^*,$$

$$q_{mn}^* = d(z) \cdot q_{m-1,n+1}^*.$$

However, this would imply $\partial p_{m-1,n+1}^* < \partial p_{mn}^*$, $\partial q_{m-1,n+1}^* < \partial q_{mn}^*$, and consequently, since $\sigma(q_{m-1,n+1}^* f - p_{m-1,n+1}^*) \geqslant m + n + 1$, $p_{m-1,n+1}^*, q_{m-1,n+1}^*$ should also be a solution of the system of order $[m, n]$. This contradicts the minimality of p_{mn}^*, q_{mn}^*. Analogously one can handle the case $\partial q_{mn}^* < n$. At last, suppose $\sigma(q_{mn}^* f - p_{mn}^*) > m + n + 1$. Then it is easy to verify that p_{mn}^*, q_{mn}^* is also the minimal solution for the problems of order $[m + 1, n]$ and $[m, n + 1]$, which contradicts the minimality of p_{mn}^*, q_{mn}^*.

(ii) (b) implies (a). Suppose that

$$\begin{aligned} p_{kl}^* &\equiv p_{mn}^*, \\ q_{kl}^* &\equiv q_{mn}^*, \end{aligned} \qquad \text{with} \quad [k, l] \neq [m, n].$$

Then, in view of the definition of minimal solution, $k \geqslant m$, $l \geqslant n$. And since $[k, l] \neq [m, n]$, $k + l > m + n$. But this would imply that

$$\sigma(q_{mn}^* f - p_{mn}^*) = \sigma(q_{kl}^* f - p_{kl}^*) \geqslant k + l + 1 > m + n + 1,$$

which contradicts the assumption that $\sigma(q_{mn}^* f - p_{mn}^*) = m + n + 1$.

(iii) (b) implies (c). Assume $H_{n+1}^{n,m} = 0$; then (2) would admit a solution with $a_{0m} = 0$. Hence $\partial p_{mn}^* < m$, which is a contradiction.

Assume $H_n^{n-1,m+1} = 0$; then (2b) would admit a solution with $b_{0n} = 0$. Hence $\partial q_{mn}^* < n$, which is a contradiction.

Assume $H_{n+1}^{n,m+1} = 0$; then there exists a solution of (2b) which also satisfies

$$\sum_{i=0}^{n} b_{0i} f_{i,m+n+1} = 0.$$

This would imply that $\sigma(q_{mn}^* f - p_{mn}^*) > m + n + 1$, which is again a contradiction.

(iv) (c) implies (b). Let $p_{mn}^* = \sum_{i=0}^{m} a_{0i}\omega_{0i}$ and $q_{mn}^* = \sum_{i=0}^{n} b_{0i}\omega_{0i}$. Since $H_n^{n-1,m+1} \neq 0$, we conclude from (2b) that rank$[m, n] = n$, and that $b_{0n} \neq 0$. Similarly $H_{n+1}^{n,m} \neq 0$ implies that $a_{0m} \neq 0$. Hence $\partial p_{mn}^* = m$ and $\partial q_{mn}^* = n$. At last $H_{n+1}^{n,m+1} \neq 0$ implies that $\sigma(q_{mn}^* f - p_{mn}^*) = m + n + 1$, which completes the proof. ∎

It is clear from this theorem that in a paranormal minimal solution table the elements are all different from each other. This however is not the case for the Newton–Padé table, as is shown by the following theorem.

THEOREM 6. *If the Newton–Padé table is paranormal and* $r_{mn} \equiv r_{m+k,n+l}$, *then* $k = l$.

Proof. Since r_{mn} and $r_{m+k,n+l}$ are paranormal, we have that

$$\partial p_{mn}^* = m, \qquad \partial q_{mn}^* = n, \qquad \text{and} \qquad \partial p_{m+k,n+l}^* = m + k, \qquad \partial q_{m+k,n+l}^* = n + l.$$

But since $r_{mn} \equiv r_{m+k,n+l}$, there exist polynomials $s_1(z)$, $s_2(z)$ such that

$$s_1(z) \cdot p_{mn}^* \equiv s_2(z) \cdot p_{m+k,n+l}^*,$$

$$s_1(z) \cdot q_{mn}^* \equiv s_2(z) \cdot q_{m+k,n+l}^*.$$

But this implies that $k = l$. ∎

From Example 1, we note, after a few calculations, that all the minimal solutions occuring in it are paranormal. Nevertheless, we have $r_{00} \equiv r_{11} = 1$ and $r_{10} \equiv r_{32} = z + 4$. Hence in a paranormal Newton–Padé table there can occur identical elements. However, by Theorem 6, these elements are restricted to lie on the same diagonal of the table.

We remark that if the sequence of interpolation points is given by

$$\{\beta_1, \beta_2, \dots, \beta_\lambda, \beta_1, \beta_2, \dots, \beta_\lambda, \beta_1, \beta_2, \dots\},$$

where the points β_i $(i = 1, 2, \dots, \lambda)$ are distinct, then the necessary and sufficient condition for the Newton–Padé table to be paranormal is that $H_{n+1}^{n,m} \neq 0$ for $n, m \in \mathbb{N}$. In the special case where $\lambda = 1$, the condition becomes $H_{n+1}^{m-n} \neq 0$

($m, n \in \mathbb{N}$). Hence a paranormal Padé table is also normal. In general this will not be true for a Newton–Padé table, as is illustrated by Theorem 6.

4. Some Remarks

(a) Concerning the Newton–Padé approximation problem the terminology is not uniquely defined. If it is considered as an approximation problem then the terminology used in this paper seems acceptable. See also [2]. However, as is indicated by Theorems 3 and 4, the Newton–Padé approximation problem is closely related to an interpolation problem. And this last point of view gives rise to the terminology "rational Hermite interpolation problem" [10], or "osculatory rational interpolation" (see, e.g., [7, 12]).

(b) The Newton–Padé problem is of interest in problems of mathematical physics [1], and also in control theory [8].

(c) Several algorithms for constructing the elements of the Newton–Padé table exist. See, e.g., [3, 4, 10].

(d) The algorithms described in [3, 4] were deduced under the condition that the Newton–Padé table was normal. We note that this existence condition can somewhat be weakened. Indeed, it is sufficient to require that the table is paranormal.

Note added in proof. As the referee pointed out, Theorem 5, which characterizes the notion of paranormality, is closely related to Theorem 3 in the paper by M. A. Gallucci and W. B. Jones, "Rational Approximations Corresponding to Newton Series (Newton–Padé Approximants)," *J. Approximation Theory* **17**, (1976), 366–392.

Acknowledgment

I want to thank L. Wuytack for his continuous interest in this work.

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